18.100 Midterm 2 Solutions

(1) (10 points)

a) Write down the definition of uniform continuity.

Solution. Let $X$ and $Y$ be metric spaces. A function $f : X \to Y$ is uniformly continuous if for every $\epsilon > 0$ there exists $\delta > 0$ such that
$$d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \epsilon$$
for all $x, y \in X$.

b) Give an example of a function that is continuous but not uniformly continuous.

Solution. For instance, $f(x) = 1/x$ defined on $(0, 1)$ furnishes a continuous function that fails to be uniformly continuous. [This can be seen as follows. Choose $\epsilon = 1$ and assume that $|f(x) - f(y)| < \epsilon = 1$ whenever $|x - y| < \delta$ for all $x, y \in (0, 1)$ and some $\delta > 0$. For $n \in \mathbb{N}$, we put $x = 1/n$ and $y = 1/2n$ so that $|x - y| = 1/2n < \delta$ for $n$ sufficiently large. But we also have $|f(x) - f(y)| = n \geq 1$, which leads to a contradiction.]

(2) (10 points)

Let $f$ be a continuous, differentiable function $f : \mathbb{R} \to \mathbb{R}$. If there is a real number $M$ such that $|f'(x)| < M$ for every $x \in \mathbb{R}$, show that $f$ is uniformly continuous.

Solution. By the mean-value theorem, we have
$$\frac{f(x) - f(y)}{x - y} = f'(\xi)$$
for all $x \neq y$ and some $\xi$ between $x$ and $y$. Thus, we infer
$$|f(x) - f(y)| \leq |f'(\xi)||x - y| < M|x - y|,$$
which, of course, is also true if $x = y$. Given $\epsilon > 0$, we choose $\delta = \epsilon/M$ to find that
$$|x - y| < \delta \implies |f(x) - f(y)| < M|x - y| = M \frac{\epsilon}{M} = \epsilon.$$
This shows uniform continuity of $f : \mathbb{R} \to \mathbb{R}$. 
(3) (10 points)
Assume that \( f : \mathbb{R} \to \mathbb{R} \) satisfies
\[
f(v + w) = f(v) + f(w)
\]
for any two real numbers \( v \) and \( w \). Assume that \( f \) is continuous at \( x = 0 \), show that \( f(z) = f(1)z \) for every \( z \in \mathbb{R} \).

**Hint:** For any \( x_0 \in \mathbb{R} \), show that \( f \) is continuous at \( x_0 \) by using \( f(x - x_0) = f(x) - f(x_0) \).

**Solution.** Clearly, \( f(0) = f(0 + 0) = f(0) + f(0) = 2f(0) \) so that \( f(0) = 0 \). Therefore the claim \( f(n) = nf(1) \) holds in particular if \( n = 0 \). Thus, by induction, we find that
\[
f(n) = f((n - 1) + 1) = f(n - 1) + f(1) = (n - 1)f(1) + f(1) = nf(1)
\]
holds for all \( n \in \mathbb{N} \). Also, \( f(0) = f(n - n) = f(n) + f(-n) \) shows that \( f(-n) = -f(n) = -nf(1) \), so that everything extends to \( \mathbb{Z} \). Moreover, we obtain \( f(nt) = nf(t) \) for all \( n \in \mathbb{Z} \) and \( t \in \mathbb{R} \). Hence, for any \( 0 \neq n \in \mathbb{Z} \), we find
\[
f(1) = f\left(\frac{n}{n}\right) = nf\left(\frac{1}{n}\right) \Rightarrow f\left(\frac{1}{n}\right) = \frac{1}{n}f(1).
\]
Combining (1) and (2), we conclude that for any rational number \( r = p/q \) where \( p, q \in \mathbb{Z} \), \( q \neq 0 \),
\[
f(r) = f\left(\frac{p}{q}\right) = pf\left(\frac{1}{q}\right) = \frac{p}{q}f(1) = rf(1).
\]
Let \( x_0 \) be any real number. To show that \( f \) is continuous at \( x_0 \), we notice
\[
f(x) - f(x_0) = f(x - x_0) \to 0 \quad \text{as} \quad x \to x_0,
\]
since \( f \) is continuous at 0. Thus \( f(x) \to f(x_0) \) whenever \( x \to x_0 \), and hence \( f \) is continuous on \( \mathbb{R} \). By continuity of \( f \) on \( \mathbb{R} \) and the fact that \( f(x) = xf(1) \) holds on the dense subset \( \mathbb{Q} \subset \mathbb{R} \), we conclude that \( f(x) = xf(1) \) holds for every \( x \in \mathbb{R} \).

(4) (10 points)
Assume that \( f \) is a differentiable function on \( (0, 1] \) with \( |f'(x)| < 1 \) for every \( x \in (0, 1] \). For every natural number \( n \geq 1 \), define
\[
a_n = f\left(\frac{1}{n}\right)
\]
and show that \( \lim_{n \to \infty} a_n \) exists. (Note that \( f(0) \) is not defined.)

**Hint:** Show that \( (a_n) \) is Cauchy.

**Solution.** The mean-value theorem and the fact that \( |f'(x)| < 1 \) give us
\[
|a_n - a_m| = \left| f\left(\frac{1}{n}\right) - f\left(\frac{1}{m}\right) \right| \leq \left| \frac{1}{n} - \frac{1}{m} \right| \leq \frac{1}{n} + \frac{1}{m}.
\]
Thus \( (a_n) \) is Cauchy, since for every \( \epsilon > 0 \) the choice of an integer \( N \geq 2/\epsilon \) leads to
\[
|a_n - a_m| < \frac{1}{n} + \frac{1}{m} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{whenever} \quad m, n \geq N.
\]
(5) (10 points)

If \( f \) is a non-negative decreasing function defined on \([0, \infty)\), \( \alpha \) is a strictly increasing function on \([0, \infty)\) and \( f \in \mathcal{R}(\alpha) \) on any interval \([0, c]\) with \( c > 0 \), prove that for any real numbers \( x \) and \( b \) satisfying \( 0 < x \leq b \) we have

\[
f(x) \leq \frac{1}{\alpha(x) - \alpha(0)} \int_0^b f \, d\alpha
\]

Solution.

\[
\int_0^b f \, d\alpha \geq \int_0^x f \, d\alpha \geq \int_0^x f(x) \, d\alpha = f(x) (\alpha(x) - \alpha(0))
\]

where we used that \( f \) is non-negative in the first inequality and that \( f \) is decreasing in the second.