18.100B Practice for the first midterm
Solutions.

Problems.

1) Let \((\mathcal{M}, d)\) be an arbitrary metric space.
   a) State the definition of a connected subset of \(\mathcal{M}\).

   **Solution.** See Definition 2.45 in Rudin.

   b) Prove that \(E \subseteq M\) is connected if and only if every non-empty proper subset has a non-empty boundary in \(E\).

   **Solution.** Notice that the equivalent statement
   
   \(E\) is separated if and only if there is a proper non-empty subset with empty boundary in \(E\),

   follows from the fact that \(A \cup B\) is a separation of \(E\) if and only if \(A\) and \(B = E \cap A^c\) have no boundary in \(E\).

2) Let \((\mathcal{M}, d)\) be an arbitrary metric space (e.g., not necessarily Euclidean space).
   a) Show that a compact subset of \(\mathcal{M}\) is necessarily closed and bounded.

   **Solution.** See Theorem 2.34 in Rudin for a proof that compact sets are closed. To prove that a compact set \(K \subseteq \mathcal{M}\) is bounded, pick any point \(p \in \mathcal{M}\) and consider the open sets \(B_n(p)\).

   These cover \(K\) (indeed, they cover \(\mathcal{M}\)), hence there is a finite subcover of \(K\) and hence \(K\) is contained in \(B_N(p)\) for large enough \(N\), i.e., \(K\) is bounded.

   b) Give an example of a metric space with a closed and bounded subset that is \(\text{NOT}\) compact.

   **Hint:** Use the discrete metric \(d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}\)

   **Solution.** Notice that any subset of a metric space with the discrete metric is closed and bounded. However, only finite subsets are compact (by a homework question), hence any infinite subset is closed, bounded, and not compact.

3) Show that \(\sqrt{2} + \sqrt{3}\) is irrational.

   **Hint:** Show that \(\sqrt{2} + \sqrt{3} \in \mathbb{Q} \implies \sqrt{2} \in \mathbb{Q}\).

   **Solution.** Let \(\sqrt{2} + \sqrt{3} = r\) then \(\sqrt{3} = r - \sqrt{2}\) and squaring both sides we get \(3 = r^2 - 2\sqrt{2} + 2\). If \(r\) is rational, then solving this equation for \(\sqrt{2}\) would give a rational expression for \(\sqrt{2}\) which we know does not exist.

4) Let \((\mathcal{M}, d)\) be an arbitrary metric space (e.g., \(\mathcal{M}\) is not necessarily complete). If \((x_n)\) and \((y_n)\) are both Cauchy sequences and \(d_n = d(x_n, y_n)\), show that \((d_n)\) is a convergent sequence of real numbers.

   **Solution.** Because \(\mathbb{R}\) is complete, we only need to show that \(d_n\) is Cauchy. Repeated use of the
triangle inequality shows that
\[ d(x_n, y_n) \leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) \implies d(x_n, y_n) - d(x_m, y_m) \leq d(x_n, x_m) + d(y_m, y_n) \]
and since the same is true reversing the roles of \( m \) and \( n \), we find
\[ |d_n - d_m| = |d(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_m, y_n). \]
Thus \((x_n)\) Cauchy and \((y_n)\) Cauchy together imply \((d_n)\) Cauchy and hence convergent.

5) Let \((\mathcal{M}, d)\) be an arbitrary metric space. If \( G \subseteq \mathcal{M} \) is open, and \( A \) is any subset of \( \mathcal{M} \), show that
\[ G \cap A = \emptyset \iff G \cap \overline{A} = \emptyset \]

**Solution.** Clearly \( G \cap \overline{A} = \emptyset \) implies \( G \cap A = \emptyset \), so suppose \( G \cap A = \emptyset \), we need to show that no point of \( G \) is a limit point of \( A \). But if \( x \in G \) then, because \( G \) is open, there is an open ball around \( x \) that stays in \( G \) and hence does not intersect \( A \), which implies that \( x \) is not a limit point of \( A \).

6) Show that if \( x, y \in \mathbb{R} \) and \( x < y \) then there exists an irrational number between \( x \) and \( y \). (You may use the existence of a rational number between \( x \) and \( y \).)

**Solution.** Let \( r \) be a rational number satisfying \( x < r < y \), we can find a large enough \( N \) so that
\[ x < r + \frac{\sqrt{2}}{N} < y \quad \left( \iff N (y - r) > \sqrt{2} \right) \]
and notice that if \( r + \frac{\sqrt{2}}{N} = q \) were rational then \( \sqrt{2} = N (q - r) \) would be rational.

An alternate proof is to note that there are uncountably many reals between \( x \) and \( y \) and there are only countably many rationals, so there must be irrationals between \( x \) and \( y \).