18.100B Practice for the final exam
Not to be turned in, just for practice.

Problems.

1) i) Let \( M \) be a metric space, state the definition of equicontinuity of a subset \( E \subseteq C (M, \mathbb{R}) \).
   
   ii) Show that if \( E \subseteq C (M, \mathbb{R}) \) is compact, then it is equicontinuous. (You may not use the Arzela-Ascoli theorem.)

2) If \( S \subseteq \mathbb{R}^n \), show that the collection of isolated points of \( S \) is countable.

3) i) Prove that if \( M \) and \( N \) are metric spaces and \( g : M \rightarrow N \) is a uniformly continuous function, then whenever \( (x_n) \subseteq M \) is Cauchy, the sequence \( (g(x_n)) \) is Cauchy.
   
   ii) Let \( M \) and \( N \) be metric spaces, let \( A \subseteq M \) and let \( \overline{A} \subseteq M \) denote the closure of \( A \). If \( N \) is complete and \( h : A \rightarrow N \) is uniformly continuous, prove that there is a unique continuous function \( \tilde{h} : \overline{A} \rightarrow N \) such that \( \tilde{h}(a) = h(a) \) for every \( a \in A \).

4) Assume \( f : (a, b) \rightarrow \mathbb{R} \) has derivative at every point in \( (a, b) \). Let \( c \in (a, b) \) and assume that
   
   \[ \lim_{x \to c} f'(x) \]
   
   exists and is finite. Prove that the value of this limit must be \( f'(c) \).

5) Assume \( f, g, \) and \( h \) are real-valued functions defined on \( [0, 1] \) and \( g \geq 0 \) is in \( \mathcal{R}(x) \).
   
   i) Prove that if \( f \) is continuous, there exists \( w \in [0, 1] \) such that
   
   \[ \int_0^1 f(t) g(t) \, dt = f(w) \int_0^1 g(t) \, dt \]
   
   Hint: Use the intermediate value theorem.

   ii) Prove that if \( h \) is monotone increasing (not necessarily continuous), there exists \( z \in [0,1] \) such that
   
   \[ \int_0^1 h(t) g(t) \, dt = h(0) \int_0^z g(t) \, dt + h(1) \int_z^1 g(t) \, dt \]
   
   Hint: Use the intermediate value theorem, but make sure to justify continuity.

6) Let \( S = \{n_1, n_2, \ldots\} \) denote the collection of those positive integers that do not involve the digit 3 in their decimal representation. (For example, \( 7 \in S \), but \( 131 \notin S \).
   
   Show that \( \sum \frac{1}{m_k} \) converges and has sum less than 90.
   
   Hint: If \( m \) has \( \ell \) digits, then \( \frac{1}{m} \leq \frac{1}{10^\ell} \). How many elements of \( S \) have \( \ell \) digits?
7) Assume that \((g_n)\) is a sequence of real-valued functions defined on \(T \subseteq \mathbb{R}\) satisfying \(g_{n+1}(x) \leq g_n(x)\) for each \(x \in T\) and \(n \in \mathbb{N}\), and suppose that \(g_n \to 0\) uniformly on \(T\). Show that

\[
\sum_{n=1}^{\infty} (-1)^{n+1} g_n(x)
\]

converges uniformly on \(T\).

8) Consider a continuous function \(f : [0, \infty) \to \mathbb{R}\). For each \(n\) define the continuous function \(f_n : [0, \infty) \to \mathbb{R}\) by \(f_n(x) = f(x^n)\). Show that the set of continuous functions \(\{f_1, f_2, \ldots\}\) is equicontinuous on some interval containing \(x = 1\) if and only if \(f\) is a constant function.

9) Define, for any \(z \in \mathbb{R}\), the exponential function by

\[
\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}.
\]

i) Prove that \(\exp : \mathbb{R} \to \mathbb{R}\) is a continuous function.

ii) Use the binomial theorem

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^n y^{n-k}
\]

to prove \(\exp(z + w) = \exp(z) \exp(w)\). Be sure to justify your steps.

iii) Prove that \(\exp'(z) = \exp(z)\). Be sure to justify your steps.