18.100B Extra practice for the final exam
Not to be turned in, just for practice.

Problems.

1) Let \( \mathcal{M} \) be a metric space and suppose there exists an \( \varepsilon > 0 \) such that the closure of \( B_\varepsilon (p) \subseteq \mathcal{M} \) is compact for all \( p \in \mathcal{M} \). Show that \( \mathcal{M} \) is complete.

2) Let \( E = \{0\} \cup \{ \frac{1}{n} : n \in \mathbb{N} \} \). Prove that \( E \) is compact using the definition (without using the Heine-Borel theorem).

3) Warning: long problem. Let \( \ell_2 \) be the set of all sequences of complex numbers \( (a_i) \) such that \( \sum |a_i|^2 < \infty \). Define a metric \( d \) on \( \ell_2 \) as follows:

\[
d ((a_i), (b_i)) := \sqrt{\sum_{i=1}^{\infty} |a_i - b_i|^2}
\]

Show that this \( d \) defines a metric on \( \ell_2 \). (Hint: Use Cauchy-Schwartz, don’t forget to show that \( d ((a_i), (b_i)) < \infty \) for \( (a_i), (b_i) \in \ell_2 \).) Show moreover that \( \ell_2 \) under this metric is complete. Is the set \( B_1 (0) \) compact?

4) Suppose that \( f : X \to X \) is a continuous function satisfying \( |f(x) - f(y)| \leq C|x - y| \). Pick a point \( x \in X \) and define a sequence \( (x_n) \) by \( x_0 = x, x_{n+1} = f(x_n) \). Prove that \( (x_n) \) is a Cauchy sequence. If the limit exists, say \( \overline{x} = \lim x_n \), show that it satisfies \( f(\overline{x}) = \overline{x} \).

5) Let \( (f_n) \) be an increasing sequence of continuous and non-negative functions on \( \mathbb{R} \) which converges pointwise to a continuous limit function \( f \). Assume that the following limit exists:

\[
\lim_{n \to \infty} \lim_{b \to \infty} \int_{-b}^{b} f_n(x) \, dx.
\]

a) Show that for any \( b > 0 \),

\[
\int_{-b}^{b} f(x) \, dx = \lim_{n \to \infty} \int_{-b}^{b} f_n(x) \, dx
\]

b) Show that the limit on the left exists and satisfies

\[
\lim_{b \to \infty} \int_{-b}^{b} f(x) \, dx \leq \lim_{n \to \infty} \lim_{b \to \infty} \int_{-b}^{b} f_n(x) \, dx
\]

c) Show that

\[
\lim_{b \to \infty} \int_{-b}^{b} f(x) \, dx = \lim_{n \to \infty} \lim_{b \to \infty} \int_{-b}^{b} f_n(x) \, dx
\]

6) Define \( \tan x = \frac{\sin x}{\cos x} \) for \( x \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \).
i) Show that \( \tan(x) \) is monotonically increasing on \((-\frac{\pi}{2}, \frac{\pi}{2})\). Conclude that the inverse function, denoted \( \arctan \), exists.

ii) Show that \( \arctan'(x) = \frac{1}{1+x^2} \).

iii) Show that

\[
\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}
\]

for \( |x| < 1 \). (Hint: First find a power series for \( \arctan'(x) \))

iv) Show that \( \arctan 1 = \frac{\pi}{4} \)

v) Prove that

\[
\frac{\pi}{4} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}
\]