A similarity transformation is defined as

\[ v^{-1} \cdot A \cdot v = B \]

where B is designated the similarity transform of A by X and A and B are conjugates of each other. A complete set of operators that are conjugate to one another is called a class of the group.

Let’s determine the classes of the group defined by E, C₃, C₃², \( \alpha_v \), \( \alpha_v' \), \( \alpha_v'' \) ... the analysis is facilitated by the construction of a multiplication table:

<table>
<thead>
<tr>
<th></th>
<th>E</th>
<th>C₃</th>
<th>C₃²</th>
<th>( \alpha_v )</th>
<th>( \alpha_v' )</th>
<th>( \alpha_v'' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td>E</td>
<td>C₃</td>
<td>C₃²</td>
<td>( \alpha_v )</td>
<td>( \alpha_v' )</td>
<td>( \alpha_v'' )</td>
</tr>
<tr>
<td>C₃</td>
<td>C₃</td>
<td>E</td>
<td>C₃²</td>
<td>( \alpha_v' )</td>
<td>( \alpha_v'' )</td>
<td>( \alpha_v )</td>
</tr>
<tr>
<td>C₃²</td>
<td>C₃²</td>
<td>E</td>
<td>C₃</td>
<td>( \alpha_v'' )</td>
<td>( \alpha_v )</td>
<td>( \alpha_v' )</td>
</tr>
<tr>
<td>( \alpha_v )</td>
<td>( \alpha_v )</td>
<td>( \alpha_v'' )</td>
<td>( \alpha_v' )</td>
<td>E</td>
<td>C₃²</td>
<td>C₃</td>
</tr>
<tr>
<td>( \alpha_v' )</td>
<td>( \alpha_v' )</td>
<td>( \alpha_v'' )</td>
<td>( \alpha_v )</td>
<td>C₃</td>
<td>E</td>
<td>C₃²</td>
</tr>
<tr>
<td>( \alpha_v'' )</td>
<td>( \alpha_v'' )</td>
<td>( \alpha_v )</td>
<td>( \alpha_v' )</td>
<td>C₃²</td>
<td>C₃</td>
<td>E</td>
</tr>
</tbody>
</table>

\[ E^{-1} \cdot C₃ \cdot E = E \cdot C₃ \cdot E = C₃ \]
\[ C₃^{-1} \cdot C₃ \cdot C₃ = C₃² \cdot C₃ \cdot C₃ = C₃² \cdot C₃² = C₃ \]
\[ (C₃²)^{-1} \cdot C₃ \cdot C₃² = C₃ \cdot C₃ \cdot C₃² = C₃ \cdot E = C₃ \]
\[ \alpha_v^{-1} \cdot C₃ \cdot \alpha_v = \alpha_v \cdot C₃ \cdot \alpha_v = \alpha_v \cdot \alpha_v' = C₃² \]
\[ (\alpha_v')^{-1} \cdot C₃ \cdot \alpha_v' = \alpha_v' \cdot C₃ \cdot \alpha_v' = \alpha_v' \cdot \alpha_v'' = C₃² \]
\[ (\alpha_v'')^{-1} \cdot C₃ \cdot \alpha_v'' = \alpha_v'' \cdot C₃ \cdot \alpha_v'' \cdot \alpha_v = C₃² \]

\[ \therefore C₃ \text{ and } C₃² \text{ form a class} \]
Performing a similar analysis on \( \alpha_v \) will reveal that \( \alpha_v, \alpha_v', \) and \( \alpha_v'' \) form a class, and \( E \) is in a class by itself. Thus there are three classes: \( E, (C_3, C_3^2), (\alpha_v, \alpha_v', \alpha_v'') \).

Additional properties of transforms and classes are:
1) no operator occurs in more than one class
2) order of all classes must be integral factors of the order of the group
3) in an Abelian group, each operator is in a class by itself

Similarity transformations give rise to irreducible representations, which lead to a useful tool in group theory – the character table. The general strategy for determining irreducible representations is as follows: \( A, B \) and \( C \) are matrix representations of symmetry operations of an arbitrary basis set (i.e., elements on which symmetry operations are performed). There is some similarity transform operator \( \nu \) such

\[
A' = \nu^{-1} \cdot A \cdot \nu \\
B' = \nu^{-1} \cdot B \cdot \nu \\
C' = \nu^{-1} \cdot C \cdot \nu
\]

where \( \nu \) uniquely produces block-diagonalized matrices—matrices possessing square arrays along the diagonal and zeros outside the blocks.

Matrices \( A, B, \) and \( C \) are reducible. Sub-matrices \( A_i, B_i \) and \( C_i \) obey the same multiplication properties as \( A, B \) and \( C \). If application of the similarity transform does not further block-diagonalize \( A', B' \) and \( C' \), then the blocks are irreducible representations. The character is the sum of the diagonal elements of their irr. rep.
Continuing with our exemplary group: E, C₃, C₃², σᵥ, σᵥ', σᵥ''... let's define the arbitrary basis ... a triangle

The basis set is described by the triangles vertices, points A, B and C. The transformation properties of these points under the symmetry operations of the group are:

\[
E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \sigma_v = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}
\]

\[
C_3 = \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \sigma_{v'} = \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

\[
C_3^2 = \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \sigma_{v''} = \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}
\]

These matrices are not block-diagonalized ... however a suitable similarity transformation will accomplish the task,

\[
v = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}, \quad v^* = \frac{1}{\sqrt{6}} \begin{bmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}
\]
Applying the similarity transformation with $C_3$ as the example,

$$v^{-1} \cdot C_3 \cdot v = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} 1 \cdot \frac{1}{\sqrt{3}} & 1 \cdot \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} \cdot \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \cdot \frac{1}{\sqrt{6}} \\ 0 \cdot \frac{1}{\sqrt{2}} & 0 \cdot \frac{1}{\sqrt{2}} \end{bmatrix} = C_3^*$$

if $v^{-1} \cdot C_3 \cdot v$ is applied again... no further block diagonalization and same trace will be obtained... an irreducible representation

The similarity transformation applied to other reducible representations yields:

$$v^{-1} \cdot E \cdot v = E^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \quad v^{-1} \cdot C_3^2 \cdot v = C_3^2* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$v^{-1} \cdot \alpha_y \cdot v = \alpha_y^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \quad v^{-1} \cdot \alpha_y'' \cdot v = \alpha_y^{''*} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$v^{-1} \cdot \alpha_y' \cdot v = \alpha_y^{''*} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & 1 \end{bmatrix}$$

As above, the block-diagonalized matrices do not further reduce under re-application of the similarity transform... all are irreducible representations
Thus a 3×3 reducible representation, $\Gamma_{\text{red}}$, has been decomposed under a similarity transformation into a 1 (1×1) and 1 (2×2) block-diagonalized irreducible representation, $\Gamma_{\text{irr}}$. The traces (i.e. sum of diagonal matrix elements) of the $\Gamma_{\text{irr}}$'s under each operation yields the characters of the representation. Taking the traces:

<table>
<thead>
<tr>
<th></th>
<th>$E$</th>
<th>$C_3$</th>
<th>$C_3^2$</th>
<th>$\sigma_v$</th>
<th>$\sigma_v'$</th>
<th>$\sigma_v''$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\Gamma_2$</td>
<td>2</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Note: characters of operators in the same class are identical

This collection of characters for a given irreducible representation, under the operations of a group is called a character table. Thus from a completely arbitrary basis, a character table is born.