Capstan—a mechanical amplifier

A schematic diagram of a basic capstan and a force diagram for a small segment of the rope are shown in the figures.

\[
F_{\text{normal}} = F \sin \Delta \theta/2 + (F+\Delta F) \sin \Delta \theta/2
\]

In the limit of small angles

\[
F_{\text{normal}} = F \, d\theta/2 + (F+dF) \, d\theta/2
\]
Assuming continuous slip and Coulomb friction between rope and drum,

\[ dF = \mu F_{\text{normal}} = \mu \frac{2F + dF}{2} \, d\theta \]

\[ dF = F \, d\theta \text{ or } d \ln F = \mu \, d\theta \]

Integrating from 0 to \( \theta \)

\[ F_{\text{out}} = e^{\mu \theta} \, F_{\text{control}} \]

Note that this relation is only valid if \( \omega r \geq v_{\text{control}} \)

From continuity: \( v_{\text{control}} = v_{\text{out}} \)

Torque required of capstan drive:

\[ \tau = (F_{\text{out}} - F_{\text{control}}) \, r = (e^{\mu \theta} - 1) \, r \, F_{\text{control}} \]

Power dissipated:

\[ P_{\text{dissipated}} = \tau \, \omega + F_{\text{control}} \, v_{\text{control}} - F_{\text{out}} \, v_{\text{out}} \]

\[ P_{\text{dissipated}} = (e^{\mu \theta} - 1) \, r \, F_{\text{control}} \, \omega - (e^{\mu \theta} - 1) \, F_{\text{control}} \, v_{\text{control}} \]

Note that \( P_{\text{dissipated}} \geq 0 \) with \( P_{\text{dissipated}} = 0 \) if \( \omega r = v_{\text{control}} \)

A bond graph follows:
This is a three-port resistor.

Typical boundary conditions result in the following causality.

Constitutive equations are

\[
\begin{bmatrix}
F_{\text{out}} \\
\tau \\
\nu_{\text{control}}
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & e^{\mu \theta} \\
0 & 0 & (e^{\mu \theta} - 1)r \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\nu_{\text{out}} \\
\omega \\
F_{\text{control}}
\end{bmatrix}
\]