HAMILTON-JACOBI THEORY

GOAL:

Find a particular canonical transformation such that the “new” Hamiltonian is a function only of the “new” momenta.

MATHEMATICAL PRELIMINARIES

A canonical transformation may be derived from a generating function.

Arguments of a generating function mix “old” and “new” variables.

e.g., old momenta, new displacements

$S(q^*, p)$

Differentiation yields “old” displacements and “new” momenta.

$p^* = -\frac{\partial S}{\partial q^*}$

$q = -\frac{\partial S}{\partial p}$
There are three other possible generating functions:

\[ S(p^*, q) \]
\[ q^* = \partial S / \partial p^* \]
\[ p = \partial S / \partial q \]

\[ S(q^*, q) \]
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\[ S(p^*, p) \]
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\[ q = -\partial S / \partial p \]
"Old" Hamiltonian

\[ H(q_1, ..., q_n, p_1, ..., p_n) \]

To find the required transformation to the "new" variables, use a generating function

\[ S(q_1, ..., q_n, p^*_1, ..., p^*_n) \]

from which

\[ p_j = \frac{\partial S}{\partial q_j} \]

"New" Hamiltonian

Substitute into the "old" Hamiltonian

\[ K(p^*_1, ..., p^*_n) = H(q_1, ..., q_n, \frac{\partial S}{\partial q_1}, ..., \frac{\partial S}{\partial q_n}) \]

This is a partial differential equation defining \( S(\cdot) \) as a function of \( q = [q_1, ..., q_n]^t \).

For the purpose of solving this equation, the "new" momenta \( p^* = [p^*_1, ..., p^*_n]^t \) and the "new" Hamiltonian \( K(p^*) \) may be treated as constant parameters.

\[ H(q_1, ..., q_n, \frac{\partial S}{\partial q_1}, ..., \frac{\partial S}{\partial q_n}) = \text{constant} \]

This is a special case of the Hamilton-Jacobi equation.

Its solution defines the required transformation.

Aside:
The Hamilton-Jacobi equation plays a prominent role in optimal control theory.
EXAMPLE:

A simple harmonic oscillator

\[
H(q,p) = \left(\frac{p^2}{2m} + \frac{kq^2}{2}\right) = \frac{1}{2m} (p^2 + mkq^2) = \frac{1}{2m} (p^2 + Z^2q^2)
\]

where \(Z = \sqrt{km}\)

set \(p = \partial S/\partial q\) and substitute

\[
H(q,\partial S/\partial q) = \frac{1}{2m} \left((\partial S/\partial q)^2 + Z^2q^2\right) = \text{constant} = K(p^*)
\]

\[
\partial S/\partial q = (2mK(p^*) - Z^2q^2)^{1/2}
\]

— a partial differential equation for \(S(q)\)

Choose \(K(p^*) = \omega p^*\)

where \(\omega = \sqrt{k/m}\)

\[
\partial S/\partial q = (2Zp^* - Z^2q^2)^{1/2}
\]

\[
S = \int (2Zp^* - Z^2q^2)^{1/2} dq
\]
differentiate to find $q^*$

$$q^* = \frac{\partial S}{\partial p^*} = \frac{Z dq}{\sqrt{2Zp^* - Z^2q^2}}$$

substitute $u = \frac{q}{\sqrt{2p^*/Z}}$

$$q^* = \int \frac{du}{1-u^2} = \sin^{-1}(u) = \sin^{-1}\left(\frac{q}{\sqrt{2p^*/Z}}\right)$$

$$q = \sqrt{2p^*/Z} \sin(q^*)$$

$$p = \frac{\partial S}{\partial q} = (2Zp^* - Z^2q^2)^{1/2} = \sqrt{2Zp^*(1 - \sin^2(q^*))}$$

$$p = \sqrt{2Zp^*} \cos(q^*)$$

New equations

$$\frac{dp^*}{dt} = -\frac{\partial K(p^*)}{\partial q^*} = 0$$

$$\frac{dq^*}{dt} = \frac{\partial K(p^*)}{\partial p^*} = \omega$$

thus

$$p^* = \text{constant}$$

$$q^* = \omega t + \text{constant}$$

The transformation $q = \sqrt{2p^*/Z} \sin(q^*)$ and $p = \sqrt{2Zp^*} \cos(q^*)$ integrates the differential equations