We then have

\[ \hat{H}|\tilde{1}\rangle = \hat{H}\hat{a}^\dagger|0\rangle \]  
\[ = \left( [\hat{H}, \hat{a}^\dagger] + \hat{a}^\dagger \hat{H} \right)|0\rangle \]  
\[ = \left( \hbar \omega \hat{a}^\dagger + \hat{a}^\dagger \frac{1}{2} \hbar \omega \right)|0\rangle \]  
\[ = \frac{3}{2} \hbar \omega \hat{a}^\dagger|0\rangle \]  
\[ = \frac{3}{2} \hbar \omega |\tilde{1}\rangle, \]  
(19-1)  
(19-2)  
(19-3)  
(19-4)  
(19-5)

i.e., \( |\tilde{1}\rangle = \hat{a}^\dagger|0\rangle \) is also an energy eigenstate, but with eigenenergy \( \frac{3}{2} \hbar \omega \) instead of \( \frac{1}{2} \hbar \omega \) for \( |0\rangle \). Similarly, we can show that \( |\tilde{2}\rangle = \hat{a}^\dagger|\tilde{1}\rangle \) is also an energy eigenstate, but with energy \( \frac{5}{2} \hbar \omega \) etc. Consequently, we can construct a ladder of (yet to be normalized) energy eigenstates \( |\tilde{n}\rangle \) by

\[ |\tilde{n}\rangle = (\hat{a}^\dagger)^n|0\rangle \]  
(19-6)

with

\[ E_n = \left( n + \frac{1}{2} \right) \hbar \omega. \]  
(19-7)

\( \hat{a} \) (\( \hat{a}^\dagger \)) is called the lowering (raising) operator, it lowers (raises) the energy by \( \hbar \omega \).

\[ \begin{array}{c}
\varepsilon_0 \\
\varepsilon_3 = \frac{3}{2} \hbar \omega - 15 \varepsilon > \\
\varepsilon_2 = \frac{3}{2} \hbar \omega - 12 \varepsilon > \\
\varepsilon_1 = \frac{3}{2} \hbar \omega - 10 \varepsilon > \\
\varepsilon_0 = \frac{1}{2} \hbar \omega - 10 \varepsilon > \\
\varepsilon_0
\end{array} \]

Figure I: \( \hat{a}, \hat{a}^\dagger \) are sometimes called “ladder operators” since they take us up and down the ladder of energy eigenstates.

When describing a monochromatic electromagnetic field quantum mechanically, we can associate the frequency \( \omega \) with a harmonic oscillator of that frequency. For non-interacting particles (such as photons) a state with \( n \) photons can be associated with the \( n \)-th eigenstate of the HO with \( n \). The ground state then corresponds to an
empty mode (no photons, \( n = 0 \)), however there is still a finite energy \( \frac{1}{2} \hbar \omega \) that we associate with vacuum fluctuations of the electromagnetic field. In this context, \( \hat{a}^\dagger \) and \( \hat{a} \) are called creation and annihilation operators, respectively, since they create and annihilate photons, or more generally, arbitrary non-interacting bosonic particles.

**Normalization of HO energy eigenstates**

Let us assume that the ground state \( |0\rangle \) is already chosen to be properly normalized: \( \langle 0|0 \rangle = 1 \).

*Note.* Remember that \( \langle 0|0 \rangle \) denotes \( \langle 0|0 \rangle = \int dx u_0^*(x) u_0(x) \).

**How long is the state** \( |1\rangle = \hat{a}^\dagger |0\rangle \)?

\[
\langle 1|1 \rangle = \langle \hat{a}^\dagger 0|\hat{a}^\dagger 0 \rangle = \langle 0|\hat{a}|\hat{a}^\dagger 0 \rangle = \langle 0|\hat{a}\hat{a}^\dagger |0 \rangle = \langle 0|[\hat{a},\hat{a}^\dagger] + \hat{a}^\dagger \hat{a}|0 \rangle = \langle 0|1 + \hat{a}^\dagger \hat{a}|0 \rangle \rightarrow (\hat{a}|0 \rangle = 0) = 1
\]

The state \( |1\rangle \) is already normalized, so we can write:

\[
|1\rangle = \hat{a}^\dagger |0\rangle \rightarrow \text{normalized eigenstate}
\]

**What about** \( |2\rangle = \hat{a}^\dagger |1\rangle = \hat{a}^\dagger |1\rangle \)?

\[
\langle 2|2 \rangle = \langle \hat{a}^\dagger 1|\hat{a}^\dagger 1 \rangle = \langle 1|\hat{a}\hat{a}^\dagger |1 \rangle = \langle 1|(\hat{a}^\dagger \hat{a} + 1)|1 \rangle = \langle 1|\hat{a}^\dagger |0 \rangle + 1 \rightarrow (\hat{a}|1 \rangle = |0 \rangle) = \langle 1|1 \rangle + 1 = 2
\]
Then the properly normalized second excited state is

\[ |2\rangle = \frac{1}{\sqrt{2}} |2\rangle = \frac{1}{\sqrt{2}} (\hat{a}^{\dagger})^2 |0\rangle. \tag{19-21} \]

We can show, in general, (see PS) that the length squared of the state \(|\tilde{n}\rangle = (\hat{a}^{\dagger})^{n} |0\rangle\) is \(\langle \tilde{n}|\tilde{n}\rangle = n!\). Consequently, the \(n\)-th normalized eigenstate is

\[ |n\rangle := \frac{1}{\sqrt{n!}} (\hat{a}^{\dagger})^{n} |0\rangle. \tag{19-22} \]

We can also show (see PS) that

\[ \hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \tag{19-23} \]
\[ \hat{a}^{\dagger}|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{19-24} \]

**From operators back to spatial wavefunctions**

The condition on the ground state \(|0\rangle\), \(\hat{a}|0\rangle = 0\), reads in position space using our definition of the annihilation operator,

\[ \hat{a} = \frac{\hat{x}}{x_0} + \frac{i}{p_0} \hat{p} = \sqrt{\frac{m\omega}{2 \hbar}} \hat{x} + \frac{i}{\sqrt{2\hbar m\omega}} \hat{p}, \tag{19-25} \]

\[ \hat{a} u_0(x) = \left( \sqrt{\frac{m\omega}{2 \hbar}} x + \frac{i}{\sqrt{2\hbar m\omega}} \frac{\hbar}{i} \frac{\partial}{\partial x} \right) u_0(x) = 0 \tag{19-26} \]
\[ \left( m\omega x + \frac{\hbar}{\partial x} \right) u_0(x) = 0. \tag{19-27} \]

The simple DE has the solution \(u_0(x) = ce^{-\frac{m\omega}{\hbar} x^2}\) with normalization \(1 = e^2 \frac{\pi \hbar}{m\omega}\).

Consequently, the normalized ground-state wavefunction is

\[ u_0(x) = \left( \frac{m\omega}{\pi \hbar} \right)^{\frac{1}{4}} e^{-\frac{m\omega}{\hbar} x^2}. \tag{19-28} \]

The normalized \(n\)-th eigenstate can be obtained from

\[ |n\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^{\dagger})^{n} |0\rangle \tag{19-29} \]

or

\[ u_n(x) = \frac{1}{\sqrt{n!}} \left( \sqrt{\frac{m\omega}{2 \hbar}} x - \frac{i}{\sqrt{2\hbar m\omega}} \frac{\hbar}{i} \frac{\partial}{\partial x} \right)^n u_0(x). \tag{19-30} \]
Commutators, Heisenberg uncertainty, and simultaneous eigenfunctions

The fact that \( \dot{\hat{p}} = \frac{\hbar}{i} \frac{\partial}{\partial x} \) in the position representation (or \( \dot{\hat{x}} = i\hbar \frac{\partial}{\partial p} \) in the momentum representation) implies

\[
\hat{p}\hat{x}\psi(x) = \hat{p}(x\psi(x)) \neq \hat{x}\dot{\hat{p}}\psi(x) = x(\dot{\hat{p}}\psi(x)),
\]

i.e., \( \hat{x} \) and \( \hat{p} \) do not commute. Define the difference between \( \hat{p}\hat{x} \) and \( \hat{x}\hat{p} \) as the commutator

\[
[\hat{p}, \hat{x}] = \hat{p}\hat{x} - \hat{x}\hat{p}.
\]

Here:

\[
[\hat{p}, \hat{x}] = \frac{\hbar}{i} \rightarrow \text{(c-number)}
\]

In general, \( [\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} \) is an operator. The commutator is linear.

\[
[c_1\hat{A}_1 + c_2\hat{A}_2, \hat{B}] = c_1[\hat{A}_1, \hat{B}] + c_2[\hat{A}_2, \hat{B}]
\]

Other useful relations

\[
[\hat{B}, \hat{A}] = -[\hat{A}, \hat{B}]
\]

\[
[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}
\]

Simultaneous eigenfunctions

Consider a free particle. The plane waves \( \psi(x) = e^{\pm ikx} \) are simultaneous eigenfunctions of energy with eigenvalue \( \frac{\hbar^2 k^2}{2m} \),

\[
\hat{H}e^{\pm ikx} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} e^{\pm ikx} = \frac{\hbar^2 k^2}{2m} e^{\pm ikx},
\]

and of momentum with eigenvalue \( \pm \hbar k \),

\[
\hat{p}e^{\pm ikx} = \frac{\hbar}{i} \frac{\partial}{\partial x} e^{\pm ikx} = \pm \hbar k e^{\pm ikx}.
\]

Note. If we had chosen \( \cos(kx), \sin(kx) \), these would have also been energy eigenfunctions with eigenvalue \( \frac{\hbar^2 k^2}{2m} \), but not momentum eigenfunctions.
However, since \( \cos(kx) \) and \( \sin(kx) \) are degenerate (i.e., have the same energy eigenvalue), it is possible to choose linear combinations of degenerate eigenstates \( e^{\pm ikx} = \cos(kx) \pm i\sin(kx) \) that are simultaneous eigenstates of momentum. In the potential well, on the other hand, the energy eigenstates were not simultaneous eigenstates of momentum. In general, we have:

**Theorem 19.1.** Two Hermitian operators \( \hat{A}, \hat{B} \) have a set of simultaneous eigenfunctions if and only if they commute.

**Proof.** 
"⇒" Assume a complete set \( \{u_{ab}\} \) of simultaneous eigenfunctions is found, i.e.,

\[
\hat{A}u_{ab} = au_{ab} \quad \text{(19-39)}
\]
\[
\hat{B}u_{ab} = au_{ab} \quad \text{(19-40)}
\]

\( a, b \), eigenvalues. Then \( [\hat{A}, \hat{B}]u_{ab} = (ab - ba)u_{ab} = 0 \) for all eigenfunctions \( \rightarrow [\hat{A}, \hat{B}] = 0 \).

"⇐" See Gasiorowicz, 5-4. \( \square \)

Since only an eigenstate of \( \hat{A} \) will have a definite outcome when a measurement of \( \hat{A} \) is made, this means that \( \Delta A \) and \( \Delta B \) can always be simultaneously made zero only when \( \hat{A} \) and \( \hat{B} \) commute.

**Theorem 19.2.** One can prove that in any chosen state \( \psi \),

\[
(\Delta A)^2_{\psi}(\Delta B)^2_{\psi} \geq \langle i[\hat{A}, \hat{B}] \rangle_{\psi}^2 \quad \text{(19-41)}
\]

for any two Hermitian operators \( \hat{A}, \hat{B} \).

**Proof.** see Gasiorowicz, online supplement SA. \( \square \)

For \( \hat{x}, \hat{p} \), we have

\[
(\Delta x)^2_{\psi}(\Delta p)^2_{\psi} \geq \frac{1}{4}(i\hbar)^2_{\psi} = \frac{\hbar^2}{4} \quad \text{(19-42)}
\]

where the RHS does not depend on the state \( \psi \). This is another derivation of the Heisenberg uncertainty relation \( \Delta x \Delta p \geq \frac{\hbar}{2} \).

**The Schrödinger equation in three dimensions**

\[
\hat{H}\psi(\mathbf{r}) = E\psi(\mathbf{r}) \quad \rightarrow \quad \text{SE in 3D} \quad \text{(19-43)}
\]

with

\[
\hat{p}^2 = \hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2 \quad \text{(19-44)}
\]
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\[ \hat{p} = (\frac{\hbar}{i} \frac{\partial}{\partial x}, \frac{\hbar}{i} \frac{\partial}{\partial y}, \frac{\hbar}{i} \frac{\partial}{\partial z}) \rightarrow \text{in the position representation} \quad (19-45) \]

The SE then reads \( \left( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \)

\[
\left[ -\frac{\hbar^2}{2m} \nabla^2 + V(r) \right] \psi(r) = W \psi(r) \rightarrow \text{SE in 3D} \quad (19-46)
\]

Spherically symmetric potential

If the potential is spherically symmetric, \( V(r) = V(r) \), then it is convenient to work in spherical coordinates, where we can write

\[
\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \left( \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right) \quad (19-47)
\]

We define an operator via

\[
\hat{L}^2 = -\hbar^2 \left( \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right) \quad (19-48)
\]

\( \hat{L} \) will be the operator associated with angular momentum.

\[
\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{\hat{L}^2}{\hbar^2 r^2} \quad (19-49)
\]

Since \( V(r) \) does not depend on \( \theta, \phi \), we try an ansatz.

\[
\psi(r) = R(r)Y(\theta, \phi) \quad (19-50)
\]

Then,

\[
\left[ -\frac{\hbar^2}{2m} \nabla^2 + V(r) \right] \psi(r) = \left[ -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right) + V(r) \right] R(r)Y(\theta, \phi) \quad (19-51)
\]

\[
+ \frac{L^2}{2mr^2} R(r)Y(\theta, \phi) \quad (19-52)
\]

\[
= ER(r)Y(\theta, \phi) \quad (19-53)
\]

As before, when deriving the time-independent SE, we divide by \( R(r)Y(\theta, \phi) \neq 0 \).

\[
\cdots = \left[ -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right) + V(r) \right] R(r) + \frac{1}{Y(\theta, \phi)} \frac{L^2}{2mr^2} R(r)Y(\theta, \phi) \quad (19-54)
\]

\[
= E \quad (19-55)
\]

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The LHS can only be a constant for all $\theta$, $\phi$ if the second term does not depend on $\theta$, $\phi$. We arrive at two equations:

\[
\frac{\hat{L}^2}{2mr^2} Y(\theta, \phi) = \frac{\text{const}}{2mr^2} Y(\theta, \phi) = E_L(r) Y(\theta, \phi) \tag{19-56}
\]

\[
\frac{1}{R(r)} \left[ -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right) + V(r) \right] R(r) + \frac{\text{const}}{2mr^2} = E \tag{19-57}
\]

\[
\left[ -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right) + V(r) + \frac{\text{const}}{2mr^2} \right] R(r) = ER(r) \tag{19-58}
\]

where $E_L = \frac{\text{const}}{2mr^2}$ is the energy associated with the angular dependence of the wavefunction.