\[ \langle p \rangle^* = \langle \hat{p} \rangle \] (13-1)
\[ = \left( \int dx \Psi^\dagger \left( \frac{\hbar}{i} \frac{\partial}{\partial x} \right) \Psi \right)^* \] (13-2)
\[ = \int dx \Psi \left( -\frac{\hbar}{i} \frac{\partial}{\partial x} \right) \Psi^* \] (13-3)
\[ = -\frac{\hbar}{i} \int_{-\infty}^{\infty} dx \left[ \frac{\partial}{\partial x} (\Psi \Psi^*) - \Psi^* \frac{\partial}{\partial x} \Psi \right] \] (13-4)
\[ = \int dx \Psi^* \left( \frac{\hbar}{i} \frac{\partial}{\partial x} \right) \Psi \] (13-5)
\[ = \langle \hat{p} \rangle, \] (13-6)

where again we have used integration by parts and the fact that \( \Psi \) vanishes at \( \pm \infty \). Consequently, \( \langle p \rangle = \langle \hat{p} \rangle^* \), i.e. all expectation values of \( \hat{p} \) are real. Since an eigenvalue is the expectation value for the corresponding stationary state, all eigenvalues of the momentum operator must be real. An operator whose eigenvalue are real (or equivalently, whose expectation value for all admissible wavefunctions is real) is called a **Hermitian operator**.

Physically measurable quantities are represented by Hermitian operators.

Similarly, one can show that \( \langle E \rangle^* = \langle E \rangle \) for any state, so all energy eigenvalue are real: The Hamiltonian operator \( \hat{H} \) is a Hermitian operator.

**Can we “derive” Newton’s \( F = ma \) from the SE?**

CM: \( F = -\frac{dv}{dt} = ma = \frac{dp}{dt} \) Let us calculate the expectation value of \( \frac{dp}{dt} \):

\[ \langle \frac{dp}{dt} \rangle = \frac{d}{dt} \langle p \rangle \] (13-7)
\[ = \frac{d}{dt} \int dx \Psi^\dagger(x,t) \left( \frac{\hbar}{i} \frac{\partial}{\partial x} \right) \Psi(x,t) \] (13-8)
\[ = \frac{\hbar}{i} \int dx \left( \frac{\partial \Psi^*}{\partial t} \frac{\partial \Psi}{\partial x} + \Psi^* \frac{\partial \Psi}{\partial x} \frac{\partial}{\partial t} \right) \] (13-9)
\[ = \int dx \left[ \frac{-\hbar^2}{2m} \frac{\partial^2 \Psi^*}{\partial x^2} \frac{\partial \psi}{\partial x} + V \Psi^* \frac{\partial \Psi}{\partial x} - \Psi^* \frac{\partial}{\partial x} \left( -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V \psi \right) \right] \] (13-10)
The integrand is

\[
A = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi^*}{\partial x^2} \frac{\partial \Psi}{\partial x} + V \Psi^* \frac{\partial \Psi}{\partial x} + \frac{\hbar^2}{2m} \frac{\partial}{\partial x} \left( \Psi^* \frac{\partial^2 \Psi}{\partial x^2} \right) \\
- \frac{\hbar^2}{2m} \frac{\partial \Psi^*}{\partial x} \frac{\partial^2 \Psi}{\partial x^2} - \Psi^* \left( \frac{\partial V}{\partial x} \right) \Psi - \Psi^* V \frac{\partial \Psi}{\partial x} \\
= \frac{\hbar^2}{2m} \frac{\partial}{\partial x} \left[ \Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x} \frac{\partial \Psi}{\partial x} \right] - \Psi^* \left( \frac{\partial V}{\partial x} \right) \Psi
\]

(13-11)

Again the integral over the first term vanishes since \( \Psi \to 0 \) for \( x \to \pm \infty \), and we are left with

\[
\langle \frac{dp}{dt} \rangle = \int dx \Psi^*(x, t) \left( -\frac{\partial V}{\partial x}(x) \right) \Psi(x, t) = \left\langle -\frac{dV}{dx} \right\rangle \quad (13-13)
\]

or

\[
\frac{d}{dt} \langle p \rangle = -\left\langle \frac{dV}{dx} \right\rangle = \left\langle -\frac{dV}{dx} \right\rangle \quad (13-14)
\]

It follows from the SE that the expectation values obey the classical equations of motion.

\[
m \frac{d}{dt} \langle x \rangle = \langle p \rangle \quad (13-15)
\]

\[
\frac{d}{dt} \langle p \rangle = -\left\langle \frac{dV}{dx} \right\rangle \quad (13-16)
\]

Average momentum changes due to average force

\[
-\left\langle \frac{dV}{dx} \right\rangle = -\int dx \Psi^* \left( \frac{\partial V}{\partial x} \right) \Psi = \int dx F(x) |\Psi(x, t)|^2,
\]

(13-17)

i.e. position-dependent force \( F(x) \) is weighted by probability density \( |\Psi(x, t)|^2 \) for finding the particle at position \( x \) at time \( t \). Note, however, that \( \left\langle \frac{dV}{dx} \right\rangle \neq \frac{d}{dx} \left\langle V(\langle x \rangle) \right\rangle \)

**Example 1.** Double-peaked distribution. The probability to find the particle at the average position \( \langle x \rangle \) is small, so the force there cannot be of much consequence for the particle’s motion.

**Example 2.** Force varying quickly on wavepacket scale. Classical calculation \( -\frac{d}{dx} \left\langle V(\langle x \rangle) \right\rangle \) would predict very large (and quickly varying force as \( \langle x \rangle \) changes), actual QM force
Figure I: Double-peaked particle distribution with vanishing probability to find particle at average position.

Figure II: Particle wavepacket large compared to spatial variation of the force. The time evolution of the wavepacket will depend on the value of the force averaged over the wavepacket, not just on the force at the average particle location.

\[ \langle -\frac{dV}{dx} \rangle \text{ experienced by particle is much smaller. However, if the force varies slowly compared to the size of a single wavepacket, then} \]

\[ \left\langle -\frac{dV}{dx} \right\rangle = \langle F(x) \rangle \approx F(\langle x \rangle) = -\frac{d}{d\langle x \rangle}V(\langle x \rangle) \quad \text{(13-18)} \]

This is the reason why we can treat particles in macroscopic potential usually as classical particles.

\[ \frac{d}{dt} \langle p \rangle = -\left\langle \frac{dV}{dx} \right\rangle \rightarrow \text{always true} \quad \text{(13-19)} \]
\[
\frac{d}{dt} \langle p \rangle \approx -\frac{d}{d\langle x \rangle} V(\langle x \rangle)
\]\[\rightarrow \text{Ehrenfest’s theorem}\] for slowly varying potentials
\[13-20\]

### Eigenfunctions of the momentum operator

What are the eigenfunctions \(u_p\) of the momentum operator, i.e. the eigenfunctions satisfying
\[
\hat{p}u_p = pu_p, \tag{13-21}
\]
where \(p\) is some (fixed) particular eigenvalue of \(\hat{p}\). We know that the operator \(\hat{p}\) is Hermitian, so all eigenvalues \(p\) are real. In position space, we have \(\hat{p} = \frac{i}{\bar{\hbar}} \frac{\partial}{\partial x}\) and \(\frac{i}{\bar{\hbar}} \frac{\partial}{\partial x} u_p(x) = pu_p(x), u_p(x) = Ae^{ipx/\hbar}.\) The momentum eigenfunctions are (of course) just the plane waves. Let us check the orthonormality condition for eigenstates:

\[
\int_{-\infty}^{\infty} dx u_p^*(x) u_{p'}(x) = \int dx e^{-ip'x/\bar{\hbar}} e^{ipx/\bar{\hbar}}
\]
\[= A_p^* A_p \int dx e^{i(p-p')x/\bar{\hbar}} \tag{13-22}\]
\[= \hbar A_p^* A_p \int dy e^{i(p-p')y} \tag{13-23}\]
\[= \hbar A_p^* A_p \delta(p-p')2\pi \tag{13-24}\]
\[= \hbar 2\pi A_{p'}^* A_p \delta(p-p') \tag{13-25}\]

The momentum eigenfunction are orthogonal for \(p \neq p'\), but we have a normalization problem for \(p = p'\): The Dirac delta function diverges, or equivalently, the integral
\[
\int_{-\infty}^{\infty} dx |u_p(x)|^2 = |A_p|^2 \int_{-\infty}^{\infty} dx \left| e^{ipx/\hbar} \right|^2
\]
\[= 1 \tag{13-27}\]

diverges. Before looking at possibilities to deal with normalization problem, let us calculate the expansion coefficients \(c(p)\)

\[
c(p) = \int dx A_p^* e^{-ipx/\hbar} \psi(x) = A_p^* \int dx \psi(x) e^{-ipx\hbar} \tag{13-28}\]
\[= \sqrt{2\pi\hbar} \delta(p) \]

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8.04 Quantum Physics

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We see that if we make the normalization choice \( A_p = \frac{1}{\sqrt{2\pi\hbar}} \), then the expansion coefficients \( c(p) \) into momentum eigenstates are just given by the Fourier transform

\[
u_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} \quad \text{"normalized" momentum eigenstates} \quad (13-29)
\]

\[
\psi(x) = \int_{-\infty}^{\infty} dp \phi(p) u_p(x)
\]

\[
= \frac{1}{\sqrt{2\pi\hbar}} \int dp \phi(p) e^{ipx/\hbar}
\]

The expansion into momentum eigenstates and Fourier transformation are one and the same. Since \( \psi(x) \) and \( \phi(p) \) contain the same information about the particle, we can use either one to characterize the position and motion of the particle. A more fundamental motion is the state of the particle (a state is a vector in Hilbert space), the state can be expressed (written down) in various representations (like position representation \( \psi(x) \), momentum representation \( \phi(p) \), energy representation \( c_E \)) associated with Hermitian operators (position \( \hat{x} \), momentum \( \hat{p} \), energy \( \hat{H} \)). We call \( \phi(p) \) the momentum representation of a particular state, and interpret it as the wavefunction in momentum space. The SE governs the time evolution of the wavefunction, or equivalently, the time evolution of the state of the particle in Hilbert space.

For one particle in one (three) dimensions, the Hilbert space is one- (three-) dimensional, but for \( N \) particles in three dimensions the Hilbert space is \( 3N \)-dimensional. In general, it cannot be factored into a tensor product of \( N \) three-dimensional vector space \( V_{system} \neq V_1 \otimes V_2 \otimes \cdots \otimes V_N \), or equivalently, the wavefunction for \( N \) particles does not factor into a product of wavefunctions for each particle,

\[
\Psi(r_1, r_2, \ldots, r_1, t) \neq \psi_1(r_1)\psi_2(r_2) \cdots \psi_N(r_N)
\]

In this case, when the wavefunction for an \( N \)-particle system cannot be written as a product of wavefunctions for the individual particles, i.e. when the particles do not evolve independently, we speak of an entangled state. Because of this possibility a quantum system of \( N \) particles is vastly (exponentially in \( N \)) richer than an classical system of \( N \) particles. However, in most cases we lose track of the particle-particle correlations associated with entanglement, and the system behaves quasi-classically. A quantum system that could preserve the correlations, and that could be manipulated externally, would constitute a quantum computer. A quantum computer could solve certain computation problems (only a handful have been discovered so far) exponentially faster than a classical computer. Because of the enormous size of the

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Hilbert space, certain quantum mechanical problems involving many-particle correlations (e.g. high temperature superconductivity that involves correlated motion of many electrons) are very difficult to solve or simulate on a classical computer. Now back to a single particle in one dimension . . .